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Simple Equations for Helical Vortex Wakes

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Introduction

THIS Note is concerned with the equations describing the wake of a propeller, wind turbine, or helicopter rotor in hover or vertical flight. In all cases, the simplest possible vortex structure in the far-wake (well downstream of the blades) consists of N helical "tip" and "hub" vortices. N is the number of blades and the quotation marks indicate that these vortices may have been modified by vortex merger in the near-wake close to the blades. This structure leads to a straightforward equation for the axial velocity in the far-wake which is substituted into the conservation equations for mass, momentum, angular momentum, and energy. The resulting equations for the pitch of the tip vortex compare favorably to some measurements for propellers, but less favorably for hovering rotors. The energy equation contains the thermodynamic efficiency whose physical basis is well-founded. For reasons that are not entirely clear, it appears that the efficiency should also be included in the torque equation. If this is correct then the near-wake cannot be force-free.

Development of the Equations

From the analysis of the Biot-Savart law in Ref. 1, the average axial velocity in the far-wake of a propeller containing only tip and hub vortices is

$$U_\infty = 1 + N\Gamma/2\pi p_\infty \tag{1}$$

where Γ is the maximum bound circulation of each blade, and p_∞ is the pitch of the tip vortex in the far-wake. All lengths are normalized by the blade radius and all velocities by the freestream velocity U_0. Note that U_∞ is independent of the details of the hub vortex and that Eq. (1) applies to the irrotational flow that is assumed to fill the region between the tip and hub vortices. In other words, Eq. (1) takes no account of the wake of the blades. In deriving Eq. (1), the induced velocity has been averaged in the circumferential direction, but not radially. Nevertheless, as required by irrotationality, the average velocity is independent of radius in the far-wake, which is the situation assumed in the usual one-dimensional analysis that leads, e.g., to the Betz limit for wind turbines.

If there is no pressure difference across the wake, and the radius of the tip vortex is much greater than that of the hub vortex, then the axial momentum equation can be written in conventional form as

$$C_T = 2U_\infty(U_\infty - 1)R_\infty^2 \tag{2}$$

where C_T is the thrust coefficient, and R_∞ is the radius of the far-wake. A similar analysis gives the equations expressing conservation of energy and angular momentum. In forming the latter, the angular momentum Wr, where W is the circumferential velocity and r is the radius, is replaced by $N\Gamma/2\pi$. The equations are

$$\eta C_P = U_\infty(U_\infty^2 - 1)R_\infty^2 \tag{3}$$

$$= NU_\infty\Gamma R_\infty^2 J^{-1} \tag{4}$$

where C_P is the power coefficient, and J is the advance ratio. η is the thermodynamic efficiency, which is not to be confused with η_P, the conventional propulsive efficiency for a propeller. The relation between the two efficiencies is

$$\eta_P = 2\eta(1 + U_\infty)^{-1} \tag{5}$$

It is common in thermodynamic analyses to include η in the energy equation—Eq. (3)—to account for the irreversible conversion of mechanical into internal energy by the action of viscosity, but most "aerodynamic" analyses do not do so, see, e.g., Eq. (11) of Ramachandran et al.[2] The appearance of η in the angular momentum equation—Eq. (4)—is also unusual and requires careful justification. This will be attempted below.

Comparison with Experiment

If C_T and C_P are known, then Eqs. (1–4) contain five unknowns: U_∞, Γ, p_∞, R_∞, and η.

Knowledge of one of the five renders the equations solvable for the others, and so allows an assessment of the simple wake equations. The author knows of no wind turbine wake measurements for which this can be done, but there is sufficient detail in the four-bladed propeller data of Ref. 3 for a mean pitch angle of 27 deg. C_T and C_P were obtained from Figs. 8 and 9, respectively, and R_∞ from Fig. 5b, all from Ref. 3, allowing estimates to be made for U_∞, Γ, p_∞, and η. The pitch can then be compared to the experimental results in Fig. 5a, also from Ref. 3. Combining Eqs. (3) and (4) and using Eq. (1) gives

$$p_\infty = J(1 + U_\infty)/2\pi \tag{6}$$

This suggests that the pitch at the blades p_1 is given by

$$p_1 = J(1 + U_1)/2\pi \tag{7}$$

where conservation of mass gives the velocity at the blades as $U_1 = U_\infty R_\infty^2$. Equations (6) and (7) are shown in Fig. 1 to be a reasonable fit to the data. Generally, both are an improvement on the "light-loading" approximation $p = J/\pi$, see, e.g., Eqs. (1–3) of Hess and Valerazo.[4] Equation (7) crosses the J/π line because $U_1 < 1$ for the two highest values of J (and $U_1 \approx 1$ for the third). This physical impossibility may well be a consequence of the crudeness of the present

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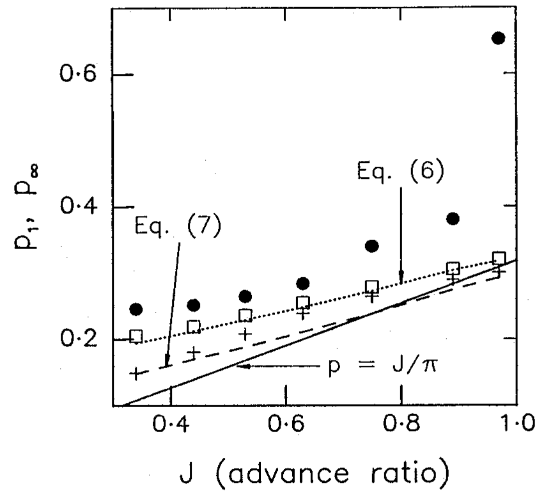

Fig. 1 Predicted and measured tip vortex pitch. ...., Eq. (6); •, Eq. (6) with η included in the denominator. - - - -; Eq. (7); experimental results from Ref. 3: p_∞, □; p_1, +.

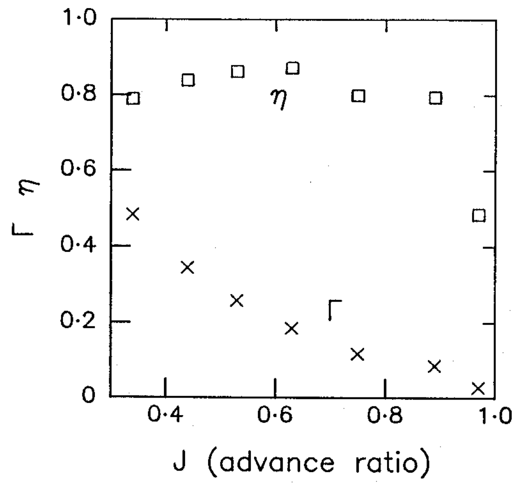

Fig. 2 Predicted blade efficiency, □, and bound circulation, ×, for the results of Ref. 3.

model. The remaining points in Fig. 1 (the solid circles) will be discussed below. Figure 2 shows the dependence on the advance ratio of the bound circulation and efficiency, neither of which can be measured directly.

Using a uniform U_1 and U_∞ is apparently reasonable for propellers, but not for hovering rotors. Figure 3 of Agarwal and Deese,[5] e.g., shows the calculated induced velocity at the blades for the experiments of Caradonna and Tung[6] increased by a factor greater than 4 from the hub to the tip. Nevertheless, the model still gives results of qualitative use. The absence of the freestream simplifies the equations even further, and the equivalent of Eqs. (2) and (5) can be combined to give

$$p_\infty = \sqrt{C_T}/2R_\infty \tag{8}$$

where, for this equation only, C_T is normalized by ΩR rather than U_0. Using the data in Fig. 7 of Ref. 6, Eq. (8) gives $p_\infty = 0.038$, whereas the figure itself indicates $p_\infty = 0.066$. Similar calculations for Fig. 8 of Langrebe[7] yield $p_\infty = 0.052$ against an experimental result of 0.079. The best agreement occurs for the tilt-rotor measurements of Felker et al.[8] where Fig. 24 shows $p_\infty = 0.102$, whereas Eq. (8) predicts $p_\infty = 0.088$. The form of Eq. (8), in particular the appearance of $\sqrt{C_T}$, bears some relation to the data correlation of Eq. (4) of Ref. 7.

Consequences of the Equations

The main consequence of the preceding analysis arises from the appearance of η in Eq. (4), which originally became apparent during the coupling of an extended form of the present model with a panel analysis of wind turbines and propellers. The extended model uses blade elements as in traditional blade element theory (BET). These are further subdivided into panels, and helical trailing vortices may form at the junction of adjacent elements as well as at the tip and hub. The wake structure is more complex than assumed here and, therefore, U_∞ depends on the radius, but Eqs. (6) and (7) are unaltered if U_∞ is now taken to be the velocity immediately beneath the tip vortex. The bound vorticity in the panel code is fixed by the Kutta condition, so that Eq. (4) is not used, but every term in the equation is calculated. It was found that the accuracy of the equation improved significantly when η was included. Removing η from Eq. (4) does not alter Eq. (7) for p_1, but Eq. (6) would now contain η in the denominator. As shown in Fig. 1, this worsens the agreement with the propeller results, but the agreement with the hovering rotor results would be improved. It is, however, possible that the disagreement with the hovering rotor results is a consequence only of the much greater variation (when compared to a propeller) of the induced velocity both at the blades and in the far-wake. This variation would cause U_1 and U_∞ to be larger than the values calculated using Eq. (2) in the manner of the previous section, and would give larger values of p_1 and p_∞. Detailed measurements of the induced velocity field would be required to settle the issue, but these are not yet available.

If the inclusion of η in Eq. (4) is correct, then only a fraction (η) of the torque produced by the blades is balanced by the angular momentum flux in the far-wake. The remaining fraction $(1 - \eta)$ must be balanced by a circumferential force acting on the trailing vortices in the near-wake. (Presumably no such force can arise from the far-wake as it would then be infinite.) Thus, we arrive at the interesting possibility that the near-wake is not force-free.

Acknowledgment

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